# Analysis I Lecture 13

Last time:

**Series** $(a_n)$ - sequence $\rightsquigarrow$

sequence of partial sums:

$(s_n)$ defined by $s_n = \sum_{i=0}^{n} a_i$.

Then we define: $\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} s_n$.

We say that series $\sum_{i=0}^{\infty} a_i$ converges if $\lim_{n \to \infty} s_n$ exists

We say that series $\sum_{i=0}^{\infty} a_i$ converges absolutely if $\sum_{i=0}^{\infty} |a_i|$ converges.

If series converges absolutely $\Rightarrow$ it converges.

## Convergence criteria:

**Theorem** If series converges absolutely, it converges.

$$\sum_{n=0}^{+\infty} |a_n| \text{ converges } \Rightarrow \sum_{n=0}^{+\infty} a_n \text{ converges}$$

Thm (Squeeze thm) Let $\sum_{i=0}^{\infty} a_i$, $\sum_{i=0}^{\infty} b_i$ be two series. then

1) If $\sum_{i=0}^{\infty} b_i$ converges and $\exists K \in \mathbb{N}$ s.t. $\forall i > K$ $\underbrace{|a_i|}_{} \leq b_i$ then $\sum_{i=0}^{\infty} a_i$ converges absolutely

2) If $\sum_{i=0}^{\infty} b_i = +\infty$ and $\exists K \in \mathbb{N}$ s.t. $\forall i > K$ $0 \leq b_i \leq a_i$ then $\sum_{i=0}^{\infty} a_i = +\infty$.

We used squeeze theorem to show that:

- $\sum_{n=0}^{\infty} \frac{1}{n} = +\infty$ diverges,

- $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

# Another application

- Series of the form $\displaystyle\sum_{n=0}^{\infty} \frac{p(n)}{q(n)}$

Polynomials

Recall $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$

- converges if $\alpha > 1$
- diverges if $\alpha \leq 1$

e.g. $\displaystyle\sum_{n=0}^{+\infty} \frac{\overbrace{n+1}^{p(n)}}{\underbrace{3n^3+n^2-2}_{q(n)}}$

$$a_n = \frac{p(n)}{q(n)}$$

$$\frac{n+1}{3n^3+n^2-2} = \frac{n\left(1+\frac{1}{n}\right)}{n^3\left(3+\frac{1}{n^2}-\frac{2}{n^3}\right)} = \overbrace{\frac{1}{n^2}\cdot\left(\frac{1+\frac{1}{n}}{3+\frac{1}{n^2}-\frac{2}{n^3}}\right)}^{a_n}$$

$\to \frac{1}{3}$ as $n \to \infty$

Since $\dfrac{1+\frac{1}{n}}{3+\frac{1}{n^2}-\frac{2}{n^3}} \longrightarrow \dfrac{1}{3}$ for some $K$ we have

that

$$0 < \frac{1+\frac{1}{n}}{3+\frac{1}{n^2}-\frac{2}{n^3}} < 1 \quad \text{for } n > K$$

Therefore $|a_n| < \dfrac{1}{n^2} \cdot 1$

$\Rightarrow$ By squeeze thm $\displaystyle\sum_{n=3}^{+\infty} a_n$ converges absolutely.

More general example:

$$\sum_{n=1}^{+\infty} \frac{n+1}{\sqrt{n^5+2n^3+4n}} \qquad a_n$$

$$a_n = \frac{n\left(1+\frac{1}{n}\right)}{n^{5/2}\cdot\sqrt{1+\frac{2}{n^2}+\frac{4}{n^4}}} = \frac{1}{n^{3/2}}\cdot\left(\frac{1+\frac{1}{n}}{\sqrt{1+\frac{2}{n^2}+\frac{4}{n^4}}}\right)$$

$$\longrightarrow 1 \text{ as } n\to\infty$$

$$\Rightarrow \quad a_n < \frac{1}{n^{3/2}}\cdot 2 \quad \text{for } n > \text{some } k$$

$\Rightarrow \quad a_n < \frac{1}{n^{3/2}} \cdot 2 \quad$ for $n >$ some $k$

Since $\frac{3}{2} > 1$ we know that

$$\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$$ converges

$\Rightarrow$ $$\sum_{n=0}^{\infty} a_n$$ converges.

By squeeze thm

Today:

More convergence criteria.

# Alternating series

<u>Definition</u> A series is called alternating if it has a form

$$\sum_{i=0}^{\infty} (-1)^i b_i \quad \text{or} \quad \sum_{i=0}^{\infty} (-1)^{i+1} b_i$$

with $b_i \geq 0 \quad \forall i$.

Alternatively series $\sum_{i=0}^{\infty} a_i$ is alternating

if $a_k \cdot a_{k+1} \leq 0 \quad \forall k \in \mathbb{N}$

Theorem (Alternating series criterion)

Let $\sum_{i=0}^{\infty} a_i$ be alternating s.t

1) $\lim_{i \to \infty} a_i = 0$

2) $|a_{i+1}| \leq |a_i| \quad \forall \; i \in \mathbb{N}$

then $\sum_{i=0}^{\infty} a_i$ converges.

Example — Alternating harmonic series.

$$\sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{1}{n}}_{a_n}$$

Alternating: $a_n \cdot a_{n+1} \le 0$ ✓

$$(-1)^n \cdot \frac{1}{n} \cdot (-1)^{n+1} \cdot \frac{1}{n+1}$$

$$= \underbrace{(-1)^{\cancel{2n+1}}}_{-1} \cdot \frac{1}{n(n+1)} < 0$$

2) $\lim_{n\to\infty} a_n = 0$ ✓

3) $|a_{n+1}| = \frac{1}{n+1} < \frac{1}{n} = |a_n|$ ✓

Therefore $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$ converges.

Remark

Alterneting sequence criterion guaranties convergence but Not ABSolute convergence.

E.g. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges but does not converge absolutely.

**Theorem** (Cauchy's criterion) Let $(x_n)$ be a sequence s.t. $\lim_{n\to\infty} \sqrt[n]{|x_n|} = \ell$.

then

1) if $\ell < 1$ $\sum_{n=0}^{\infty} x_n$ converges absolutely

2) if $\ell > 1$ $\sum_{n=0}^{\infty} x_n$ diverges

3) if $\ell = 1$ can't say anything

Example: Sequences with $\lim_{n\to\infty} \sqrt[n]{|x_n|} = 1$

- $x_n = (-1)^n$: $\sum_{n=0}^{\infty} x_n$ diverges but $(s_n)$ is bounded

$$S_n = \underbrace{1 - 1 + 1 - \dots \pm 1}_{n+1 \text{ terms}} = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

- $x_n = n$: $\sum_{n=0}^{\infty} x_n = +\infty$ and in particular diverges.

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

- $x_n = \frac{1}{n}$ $\qquad \sum_{n\geq 0}^{\infty} x_n = +\infty$ diverges

$\hookrightarrow$ $\qquad s_n$ is unbounded

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = 1$$

- $x_n = \frac{1}{n^2}$ $\qquad \sum_{n=0}^{\infty} x_n$ converges absolutelly

$\hookrightarrow$

$$\sqrt[n]{\frac{1}{n^2}} = \frac{1}{n^{2/n}} \longrightarrow 1 \text{ as } n \to \infty$$

- $x_n = \dfrac{(-1)^n}{n}$ $\Rightarrow$ $\displaystyle\sum_{n=0}^{\infty} x_n$ converges but does not converge absolutely.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1.$$

## Theorem (D'Alambert criterion)

Let $(x_n)$ be a sequence s.t.

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \rho .$$ Then

if $\rho < 1$ then $\sum_{k=0}^{\infty} x_n$ converges absolutly

if $\rho > 1$ then $\sum_{k=0}^{\infty} x_n$ diverges

if $\rho = 1$ can't say anything

Example $\displaystyle\sum_{k=0}^{\infty} \frac{k}{2^k}$

D'Alamber: $\displaystyle\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \lim_{k\to\infty} \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} =$

$\displaystyle = \lim_{k\to\infty} \frac{k+1}{k} \cdot \frac{2^k}{2^{k+1}} = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2} < 1$

$\Rightarrow$ By D'Alamber $\displaystyle\sum_{k=0}^{\infty} \frac{k}{2^k}$ converges absolutely.

$$\sum_{k=0}^{\infty} \frac{k}{2^k}$$

Cauchy: $\lim_{k\to\infty} \sqrt[k]{|x_k|} =$

$$= \lim_{k\to\infty} \sqrt[k]{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{\sqrt[k]{k}}{2} = \frac{1}{2} < 1$$

$\sqrt[k]{k} \to 1$ as $k\to\infty$

$\Rightarrow$ by Cauchy $\sum_{k=0}^{\infty} \frac{k}{2^k}$ converges absolutely.

## Example

$$x_n = \frac{n!}{n^n}$$

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

D'Alamber:

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+\cancel{1})!}{\cancel{n!}} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$n+1$

$$= \lim_{n\to\infty} \frac{n^n \cdot (n+1)}{(n+1)^{n+1}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n =$$

$$= \lim_{n\to\infty} \frac{1^n}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}} = \frac{1}{e} < 1$$

$1^n = 1 \quad \forall n$

$\Rightarrow$ by D'Alambert $\displaystyle\sum_{n=1}^{+\infty} \frac{n!}{n^n}$ converges absolutely

Example For which values of x the series $\displaystyle\sum_{k=0}^{\infty} \underbrace{\frac{x^k}{k!}}_{a_k}$ converges?

x is real number

Let's apply D'Alamber:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} =$$

$$= \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{x^k}{k!}$$ converges for any $x \in \mathbb{R}$.

Example For which $x$ the series

$$\sum_{k=0}^{+\infty} \underbrace{2^k \cdot (x-1)^k}_{a_k} \quad \text{converges?}$$

D'Alambert $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|2^{k+1} \cdot (x-1)^{k+1}|}{|2^k (x-1)^k|} =$

$$= \lim_{k\to\infty} 2 \cdot |x-1| = 2 \cdot |x-1|$$

$$2|x-1| < 1 \iff |x-1| < \frac{1}{2}$$

$$\iff -\frac{1}{2} < x-1 < \frac{1}{2}$$

$$\sum_{k=0}^{\infty} 2^k(1-x^k) \text{ converges if } \iff \boxed{\frac{1}{2} < x < \frac{3}{2}}$$

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$$2|x-1| > 1 \iff \boxed{x > \frac{3}{2} \text{ or } x < \frac{1}{2}}$$

$$\sum_{k=0}^{\infty} 2^k(1-x^k) \text{ diverges}$$

For $x = \frac{1}{2}, \frac{3}{2}$, $\lim\limits_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$

So D'Alambeer is not applicablee.

Let's Analize by hand:

$x = \frac{1}{2}$ : $$\sum_{k=0}^{\infty} 2^k \left(\frac{1}{2} - 1\right)^k = \sum_{k=0}^{\infty} (-1)^k$$ diverges

$x = \frac{3}{2}$ : $$\sum_{k=0}^{\infty} 2^k \left(\frac{3}{2} - 1\right)^k = \sum_{k=0}^{\infty} 1$$ diverges.

So we got that

For $\frac{1}{2} < x < \frac{3}{2}$ series

$$\sum_{k=0}^{\infty} 2^k (x-1)^k$$ converges

and it diverges for

other values of $x$.

# REAL FUNCTIONS OF ONE VARIABLE

Domain of f

**Definition** Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function.

1) We will define the image of f as

$$\operatorname{Im}(f) = \{ y \mid y = f(x) \text{ for some } x \in D \}$$

Definition (1) $f$ is bounded above (or below) if there exists $C \in \mathbb{R}$ s.t. $\forall\ x \in D$ we have $f(x) < C$ $\left(f(x) > C\right)$.

This is equivalent to $\operatorname{Im}(f)$ is bounded above (or below).

As usual, we say that $f$ is bounded if it is bounded both below and above.

Definition

- $f$ is increasing (decresing)

if $\forall\, x, y \in D$ s.t. $x < y$

$f(x) \leq f(y)$ $\big( f(x) \geq f(y) \big)$

- $f$ is stricly increasing (stricly decresing)

if $\forall\, x, y \in D$ s.t $x < y$

$f(x) < f(y)$ $\big($ or $f(x) > f(y) \big)$

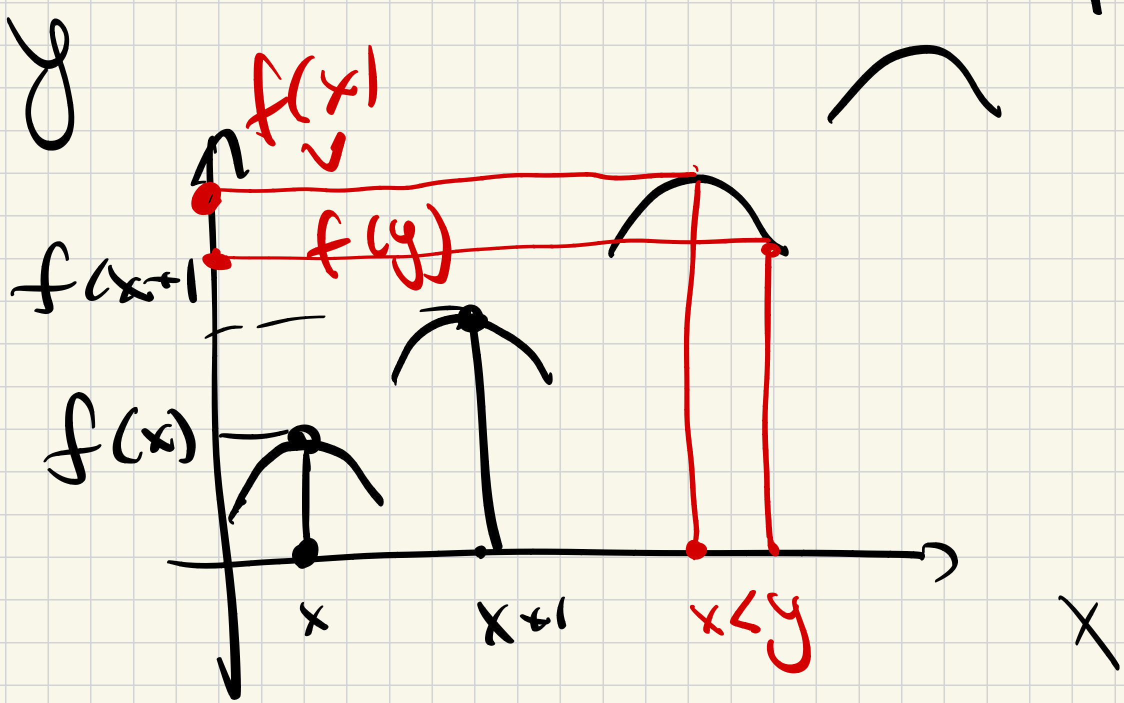
Note!

Not enough to check

$f(x+1) > f(x)$ !

Here $f(x+1) > f(x)$

But $f$ is not increasing



<u>Definition</u> $f$ is (stricly) monotone if it is either (stricly) increasing

or (stricly) decreasing.

**Definition** $f$ is called periodic

if $\exists T > 0$ s.t. $\forall x \in D$ s.t.

$x + T \in D$ we have $f(x) = f(x+T)$